Complex analysis: limits and continuity.

Thursday, September 14, 2023 11:00 AM

Limits and Continuity: a Short review.

Same as for $\mathbb{R}^2$!

Def. Let f be a function defined on a set $K \subseteq \mathbb{C}$.
f has a limit A as $z \to z_0$ if

$$\forall \varepsilon > 0 \ \exists \delta > 0 : 0 < |z - z_0| < \delta, \ z \in K \Rightarrow |f(z) - A| < \varepsilon.$$

Properties. 1) If the limit exists it is unique provided z_0 is a limit point of K

$$(\forall \delta > 0 : \ B(z_0, \delta) \cap (K \setminus \{z_0\}) \neq \emptyset).$$

2) $\lim\limits_{z \to z_0} (f(z) + g(z)) = \lim\limits_{z \to z_0} f(z) + \lim\limits_{z \to z_0} g(z)$ (if both exist)

3) $\lim\limits_{z \to z_0} (f(z) \times g(z)) = \lim\limits_{z \to z_0} f(z) \times \lim\limits_{z \to z_0} g(z)$ (if both exist)

4) $\lim\limits_{z \to z_0} f(z) = A \Leftrightarrow$ $\lim\limits_{z \to z_0} \operatorname{Re} f(z) = \operatorname{Re} A$, $\lim\limits_{z \to z_0} \operatorname{Im} f(z) = \operatorname{Im} A$.

5) $\lim\limits_{z \to z_0} f(z) = A \Rightarrow \lim\limits_{z \to z_0} \overline{f(z)} = \overline{A}$.

Proof. 1, 2, 3 – as in real case.

5. $\Leftarrow$ $|f(z) - A| = |\overline{f(z)} - \overline{A}|$.

4 $\Leftarrow$ 5+2, since $\operatorname{Re} f(z) = \dfrac{f(z) + \overline{f(z)}}{2}$

$\operatorname{Im} f(z) = -\frac{i}{2}(f(z) - \overline{f}(z))$.

Important property: $K_1, K_2 \subset K$. Let $\lim\limits_{\substack{z \to z_0 \\ z \in K}} f(z) = A$.

Then $\lim\limits_{\substack{z \to z_0 \\ z \in K_1}} f(z) = \lim\limits_{\substack{z \to z_0 \\ z \in K_2}} f(z) = A$.

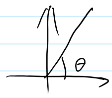
Proof. Fix $\varepsilon > 0$, $\exists \delta > 0 \ldots$
$z \in K_1$, $|z - z_0| < \delta \Rightarrow z \in K$, $|z - z_0| < \delta \Rightarrow |f(z) - A| < \varepsilon$ ∎

<u>Corollary.</u> $K_1, K_2 \subset K$. $\lim\limits_{\substack{z \to z_0 \\ z \in K_1}} f(z) \neq \lim\limits_{\substack{z \to z_0 \\ z \in K_2}} f(z) \Rightarrow$

$\lim\limits_{z \to z_0} f(z)$ does not exist.

<u>Easy and important example:</u>

$\lim\limits_{h \to 0} \dfrac{\overline{h}}{h}$ does not exist!



On ray L_θ: $h = |h| \operatorname{cis} \theta$

On L_θ: $\dfrac{\overline{h}}{h} = \dfrac{|h| \operatorname{cis}(-\theta)}{|h| \operatorname{cis}(\theta)} = \operatorname{cis}(-2\theta)$ – different on different rays!

<u>Continuous functions.</u>

As usual: f is continuous at z_0 if $\lim\limits_{\substack{z \to z_0 \\ z \in K}} f(z) = f(z_0)$.

<u>Remark</u> All of this can be done at ∞, but we need to use <u>spherical metric</u>:

$\lim\limits_{z \to z_0} f(z) = \infty \iff \lim\limits_{z \to z_0} d(f(z), \infty) = 0$

$\Updownarrow$

$\lim\limits_{z \to z_0} \dfrac{1}{|f(z)|} = 0 \iff \lim\limits_{z \to z_0} \dfrac{1}{\sqrt{1 + |f(z)|^2}} = \lim\limits_{z \to z_0} \dfrac{1}{|f(z)|} \cdot \dfrac{|f(z)|}{\sqrt{1 + |f(z)|^2}} = 0$

$\lim\limits_{z \to \infty} f(z) = A \iff \forall \varepsilon > 0\ \exists \delta > 0 : 0 < d(z, \infty) < \delta \Rightarrow |f(z) - A| < \varepsilon$

$\Updownarrow$

$\forall \varepsilon > 0\ \exists \delta > 0 : \dfrac{1}{\ } < \delta \Rightarrow |f(z) - A| < \varepsilon$

$z \to \infty$

$$\Updownarrow$$

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \ \frac{1}{\sqrt{1+|z|^2}} < \delta \Rightarrow |f(z) - A| < \varepsilon$$

$$\Updownarrow \quad \left(\begin{array}{l} \text{if } \delta < \frac{1}{2}, \ \frac{1}{\sqrt{1+|z|^2}} < \delta \Rightarrow \frac{1}{|z|} < 2\delta \\ \text{For any } \delta, \ \frac{1}{|z|} < \delta \Rightarrow \frac{1}{\sqrt{1+|z|^2}} < \delta \end{array} \right)$$

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \ \frac{1}{|z|} < \delta \Rightarrow |f(z) - A| < \varepsilon$$

Important (and easy) observation: if $z_0 \neq \infty$ then

$$\lim_{z \to z_0} |z - z_0| = 0 \iff \lim_{z \to z_0} d(z, z_0) = \lim_{z \to z_0} \frac{|z - z_0|}{\sqrt{1+|z|^2}\sqrt{1+|z_0|^2}} = 0.$$